Exercise 1. Show that the dipole $\delta_0' : \mathscr{D}(\mathbb{R}) \to \mathbb{R}$ defined by

$$\langle \delta_0', \varphi \rangle = -\varphi'(0)$$

is a distribution, *i.e.* en element of $\mathscr{D}'(\mathbb{R})$.

Exercise 2. Let $\Omega \subset \mathbb{R}^d$ be an open set and $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{D}'(\Omega)$ be a sequence of distributions converging towards $T \in \mathscr{D}'(\Omega)$. Show that for all multi-index $\alpha \in \mathbb{N}^d$ the sequence $\{D^\alpha T_n\}_{n\in\mathbb{N}}$ converges to $D^\alpha T$.

Exercise 3. Let $\{S_n\}_{n\in\mathbb{N}^*}, \{T_n\}_{n\in\mathbb{N}^*} \subset \mathscr{D}'(\mathbb{R})$ be the sequences given by

$$T_n = \delta_{\frac{1}{n}} \quad \text{and} \quad S_n = n(T_n - T_{2n}).$$

Compute the limit in $\mathscr{D}'(\mathbb{R})$ of $\{S_n\}_{n\in\mathbb{N}^*}$ and of $\{T_n\}_{n\in\mathbb{N}^*}$ as $n \to \infty$.

Exercise 4.

1. Compute the distribution $e^x \cdot \delta_0''$.
2. Let $a, b > 0$. Compute the distributional derivative

$$f_{a,b} = H(x)\log|a\,x| + H(-x)\log|b\,x|,$$

where H is the Heaviside function.

Exercise 5. Let $\{\varepsilon_n^\pm\}_{n\in\mathbb{N}} \subset (0,\infty)$ be two sequences of positive numbers that converge to 0, *i.e.* such that

$$\lim_{n\to\infty} \varepsilon_n^\pm = 0.$$

Assume that there exists $0 < a < \infty$ such that $\dfrac{\varepsilon_n^+}{\varepsilon_n^-} \underset{n\to\infty}{\longrightarrow} a$. Show that the distribution $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{D}'(\mathbb{R})$ defined for all $\varphi \in \mathscr{D}(\mathbb{R})$ by

$$T_n(\varphi) = \int_{-\infty}^{-\varepsilon_n^-} \frac{\varphi(x)}{x}dx + \int_{\varepsilon_n^+}^{\infty} \frac{\varphi(x)}{x}dx$$

converges in $\mathscr{D}'(\mathbb{R})$, and compute its limit.

Exercise 6. Show that the function $g : \mathbb{R} \to \mathbb{R}, x \mapsto \max\{x, 0\}$ is the fundamental solution of the operator $L = \dfrac{\partial^2}{\partial x^2}$, *i.e.* show that we have

$$Lg = \delta_0 \qquad \text{dans } \mathscr{D}'(\mathbb{R}).$$

Exercise 7. Let $G : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}, x \mapsto \dfrac{1}{2\pi}\log|x|$.

Show that G is a distribution on $\mathbb{R}^2$ and that G is a fundamental solution of the Laplacian, *i.e.* $\Delta G = \delta_0$ in $\mathscr{D}'(\mathbb{R}^2)$.

Hint : consider for all $\varepsilon \to 0$ the integral

$$\int_{\mathbb{R}^2\setminus\overline{B}(0,\varepsilon)} G(x)\Delta\varphi(x)dx,$$

where $\varphi \in \mathscr{D}(\mathbb{R}^2)$, and integrate by parts.